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Asymptotic convergence criteria of solutions of delayed functional differential equations

Josef Diblík

Department of Mathematics, Faculty of Electrical Engineering and Communication, Brno University of Technology, Technická 8, 616 00 Brno, Czech Republic

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Abstract

The convergence of all solutions of the scalar equation $\dot{x} = f(t, x_t)$ is studied under the main assumption that every constant is its solution. The criterion and the sufficient conditions for convergence are proved. For the considered classes of equations it is, among others, proved that all solutions are convergent if and only if there exists at least one increasing and convergent solution. As applications, criteria for some classes of linear equations as well as for nonlinear equations are obtained. Some comparisons with already known results are given too.

Keywords: Convergence of solutions; Functional differential equation

1. Introduction and main results

Let $C([a,b],\mathbb{R})$ where $a, b \in \mathbb{R}$, $a < b$, $\mathbb{R} = (-\infty, +\infty)$ be the Banach space of continuous functions mapping the interval $[a,b]$ into $\mathbb{R}$ with the topology of uniform convergence. In the case $a = -r < 0$, $b = 0$ we will denote this space as $\mathcal{C}$, that is $\mathcal{C} = C([-r,0],\mathbb{R})$ and designate the norm of an element $\varphi \in \mathcal{C}$ by $\|\varphi\| = \sup_{-r\leqslant\theta\leqslant 0} |\varphi(\theta)|$.

E-mail address: diblik@feec.vutbr.cz.

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In this paper we investigate the problem of convergence of all solutions of a retarded functional differential equation

$$\dot{x} = f(t, x_t) \tag{1}$$

where $f : \Omega \to \mathbb{R}$ is a *continuous quasibounded map* which satisfies a *local Lipschitz condition* with respect to the second argument in each compact set in Ω where Ω is an open subset in $\mathbb{R} \times \mathcal{C}$. Symbol "$\cdot$" represents the right-hand derivative.

If $\sigma \in \mathbb{R}$, $A \geqslant 0$ and $x \in C([\sigma - r, \sigma + A], \mathbb{R})$, then for each $t \in [\sigma, \sigma + A]$ we define $x_t \in \mathcal{C}$ by means of relation $x_t(\theta) = x(t + \theta)$, $\theta \in [-r, 0]$.

In accordance with [12] a function $x(t)$ is said to be a *solution of Eq.* (1) *on* $[\sigma - r, \sigma + A)$ with $\sigma \in \mathbb{R}$ and $A > 0$, if $x \in C([\sigma - r, \sigma + A), \mathbb{R})$, $(t, x_t) \in \Omega$ and $x(t)$ satisfies Eq. (1) for $t \in [\sigma, \sigma + A)$.

For given $\sigma \in \mathbb{R}$, $\varphi \in \mathcal{C}$, $(\sigma, \varphi) \in \Omega$, we say that $x(\sigma, \varphi)$ is a *solution of Eq.* (1) *through* (σ, φ) (or that $x(\sigma, \varphi)$ *corresponds to the initial point* σ) if there is an $A > 0$ such that $x(\sigma, \varphi)$ is a solution of Eq. (1) on $[\sigma - r, \sigma + A)$ and $x_\sigma(\sigma, \varphi) = \varphi$.

In view of the above conditions, *each element* $(\sigma, \varphi) \in \Omega$ *determines a unique solution* $x(\sigma, \varphi)$ of Eq. (1) through $(\sigma, \varphi) \in \Omega$ on its maximal interval of existence $I_{\sigma,\varphi} = [\sigma, \sigma + A)$, $A > 0$. This solution *depends continuously* on the initial data [12]. Solution $x(\sigma, \varphi)$ of Eq. (1) is *convergent* or *asymptotically convergent* if $I_{\sigma,\varphi} = [\sigma, \infty)$ and if $x(\sigma, \varphi)(t)$ has a finite limit for $t \to \infty$.

We say that Eq. (1) has *asymptotic equilibrium point* at the point σ if and only if every solution $x(\sigma, \varphi)(t)$ of Eq. (1) with $\varphi \in \mathcal{C}$ is convergent and for every $\xi \in \mathbb{R}$ there exists a solution $x^{\xi}(\sigma, \varphi)(t)$ of Eq. (1) which tends to ξ at ∞.

In the following we will use the abbreviations $\mathbb{R}^+ = (0, \infty)$, $I = [t_0 - r, \infty)$ and $I_0 = [t_0, \infty)$ with fixed $t_0 \in \mathbb{R}$.

The paper is divided in the following sections: In this section the basic assumptions and problems are formulated as well as the main results. In Section 2 some appropriate classes of linear and nonlinear equations are showed. Section 3 is devoted to the formulation of concrete convergence criteria. Proofs of results (and corresponding preliminaries) are collected in Section 4. In the proofs of the main results the topological Ważewski method is used.

In the text, several papers are cited in which problems of convergence or divergence are considered, and some comparisons are made as well. Let us point out here only several principal moments. In the known papers there are mostly given only sufficient conditions for convergence. Nevertheless, criteria of convergence, given in the present paper, can serve as a source for various sufficient convergence conditions. This is shown in Section 3.

The classes of equations considered here are different from the classes considered in papers [2] and [14]. In paper [2] the considered classes of equations are linear and quasilinear and proofs of results are based on different techniques (variation-of-constant formula and decomposition theory of linear autonomous functional differential equations), which cannot be used here. In the paper [14] the Liapunov technique and differential inequalities are applied to get sufficient convergence results for functional differential equations with infinite delay. This approach is not applicable here since we deal with equations for which the upper right-hand derivative of appropriate Liapunov function does not satisfy the necessary inequalities.

1.1. Basic assumptions

Let us assume (except for the assumptions of continuity, quasiboundedness and local Lipschitzeanity which were supposed above) the hypotheses (i)–(v) below with respect to the right hand side of Eq. (1):

(i) *For any constant* $L \in \mathbb{R}$ *and for all* $(t, x_t) \in \Omega$:

$$f(t, x_t) \equiv f(t, x_t + L).$$

(ii) *The functional* $f(t, x_t)$ *is odd with respect to the second argument for any* $(t, x_t) \in \Omega$, *i.e.*

$$f(t, -x_t) \equiv -f(t, x_t).$$

(iii) *There exists a positive constant* k_0 *such that for every* $k \geqslant k_0$ *and every* $(t, x_t) \in \Omega$ *with nonincreasing* $x_t(\theta)$ *for* $\theta \in [-r, 0]$:

$$f(t, kx_t) \geqslant kf(t, x_t).$$

(iv) *For every* $(t, \varphi_t) \in \Omega$, $(t, x_t) \in \Omega$ *with positive nonincreasing* $\varphi_t(\theta)$ *for* $\theta \in [-r, 0]$, *with* $|x(t+\theta)| < \varphi(t+\theta)$ *for* $\theta \in [-r, 0)$ *and with* $\varphi_t(0) = x_t(0)$:

$$f(t, x_t) > f(t, \varphi_t).$$

(v) *For every* $(t, \varphi_t) \in \Omega$ *with* $\varphi_t(0) > \varphi_t(\theta)$ *for* $\theta \in [-r, 0)$:

$$f(t, \varphi_t) > 0.$$

Remark 1. As follows from (ii), Eq. (1) has the trivial solution. Furthermore, it follows from (i) and (ii) that each constant is a solution of Eq. (1) and every shift of a solution by a constant gives again a solution.

1.2. Considered problems

In this paper the relations between the following affirmations (α)–(δ) will be studied:

(α) *There exists increasing and convergent solution of Eq.* (1) *on* I.

(β) *For each solution* $x(\sigma, \varphi)$ *of Eq.* (1) *with* $(\sigma, \varphi) \in \Omega$, $\sigma \geqslant t_0$ *it holds*: $I_{\sigma,\varphi} = [\sigma, \infty)$ *and* $x(\sigma, \varphi)$ *is convergent.*

(γ) *There exists decreasing and convergent function* $\nu \in C(I, \mathbb{R}) \cap C^1(I_0, \mathbb{R})$ *satisfying the inequality*

$$\dot{\nu}(t) \leqslant f(t, \nu_t), \quad t \in I_0. \tag{2}$$

(δ) *There exist a positive constant* m *and a function* $\lambda : I \to \mathbb{R}^+$ *such that*

$$\lambda(t) \geqslant -\big[I(m,\lambda)(t)\big]^{-1} \cdot f\big(t, I(m,\lambda)_t\big), \quad t \in I_0 \tag{3}$$

where $I(m,\lambda)(t) \equiv m \exp(-\int_{t_0}^{t} \lambda(s)\,\mathrm{d}s)$.

1.3. Main results

In this part the main results of the paper are formulated.

Theorem 1. *Let hypothesis* (v) *hold. Then the affirmation* (β) *implies the affirmation* (α). *If, moreover,* (ii) *holds, then the affirmation* (β) *implies the affirmation* (γ).

Theorem 2. *Let hypothesis* (i) *hold. Then the affirmations* (γ) *and* (δ) *are equivalent.*

Theorem 3. *Let hypotheses* (i)–(iv) *hold. Then the affirmation* (α) *implies the affirmations* (β)–(δ).

Theorem 4 (Main result). *Let hypotheses* (i)–(v) *hold. Then the affirmations* (α)–(δ) *are equivalent.*

Moreover, the result concerning the existence of asymptotic equilibrium and continuability of solutions, as a simple consequence of Theorem 3 and property (i), holds:

Corollary 1. *If hypotheses* (i)–(iv) *hold and if there exists an increasing and convergent solution of Eq.* (1) *on* I, *then for every solution* $x(\sigma, \varphi)$ *with* $(\sigma, \varphi) \in \Omega$, $\sigma \geqslant t_0$ *of Eq.* (1) *it holds*: $I_{\sigma,\varphi} = [\sigma, \infty)$. *Moreover, for every* $\xi \in \mathbb{R}$, *there exists a* nonconstant *solution* $x = x^{\xi}(t)$ *of Eq.* (1) *on* I *such that*

$$\lim_{t\to\infty} x^{\xi}(t) = \xi,$$

i.e. Eq. (1) *has the asymptotic equilibrium at any point* $\sigma \geqslant t_0$.

2. Some classes of appropriate equations

In this part we will show that there exist nonempty classes of linear as well as nonlinear equations satisfying all the hypotheses (i)–(v) from Section 1.1. (For linear equations the hypotheses (ii), (iii) are satisfied automatically.) It can be verified easily that to these classes there belong, e.g., the following equations:

Equation 1.

$$\dot{x}(t) = \beta(t)\left[x(t) - \sum_{i=1}^{n} \beta_i(t) x\big(t - r_i(t)\big)\right], \quad t \in I_0 \tag{4}$$

with $\beta, \beta_j \in C(I_0, \mathbb{R}^+)$, $r_j \in C(I_0, (0, r])$, $j = 1, \dots, n$ and with $\sum_{i=1}^{n} \beta_i(t) \equiv 1$.

Equation 2.

$$\dot{x}(t) = \int_{t-\tau(t)}^{t} \beta(t, s)\big[x(t) - x(s)\big]\,\mathrm{d}s, \quad t \in I_0$$

with $\beta \in C(I_0 \times I, \mathbb{R}^+)$ and $\tau \in C(I_0, (0, r])$.

Equation 3.

$$\dot{x}(t) = \beta(t) \arctan\big(x(t) - x\big(t - \tau(t)\big)\big), \quad t \in I_0$$

with $\beta \in C(I_0, \mathbb{R}^+)$ and $\tau \in C(I_0, (0, r])$.

Equation 4.

$$\dot{x}(t) = \frac{\beta(t)[x(t) - x(t - \tau(t))]}{\gamma(t) + \delta(t)|x(t) - x(t - \tau(t))|^{\alpha}}, \quad t \in I_0 \tag{5}$$

with $\alpha \in [0, 1)$, $\alpha = \text{const}$, $\beta, \gamma, \delta \in C(I_0, \mathbb{R}^+)$ and $\tau \in C(I_0, (0, r])$.

Remark 2. Let us briefly explain the motivation for the investigation of classes of equations of the form (4). Let us consider the linear differential equation with delayed arguments, used in many investigations:

$$\dot{y}(t) = \alpha(t) y(t) - \sum_{i=1}^{n} \alpha_i(t) y\big(t - r_i(t)\big) \tag{6}$$

with $\alpha \in C(I_0, \mathbb{R})$, $\alpha_j \in C(I_0, \mathbb{R}^+)$ and $r_j \in C(I_0, (0, r])$, $j = 1, \dots, n$. Suppose the existence of a positive solution $y = \varphi$ of Eq. (6) on I. Then the substitution $y(t) = \varphi(t) \cdot x(t)$ gives a new equation with respect to x:

$$\begin{aligned}\dot{x}(t) = {} & \left(\varphi^{-1}(t) \sum_{i=1}^{n} \alpha_i(t) \varphi\big(t - r_i(t)\big)\right) \\ & \times \left[x(t) - \sum_{i=1}^{n} \left(\frac{\alpha_i(t)\varphi(t - r_i(t))}{\sum_{j=1}^{n} \alpha_j(t)\varphi(t - r_j(t))}\right) x\big(t - r_i(t)\big)\right].\end{aligned}$$

This equation (where φ^{-1} is continuous and positive on I due to the definition of φ) is an equation of the type (4) with

$$\beta(t) \equiv \varphi^{-1}(t) \sum_{i=1}^{n} \alpha_i(t)\varphi\bigl(t - r_i(t)\bigr) \quad \text{and}$$

$$\beta_i(t) \equiv \frac{\alpha_i(t)\varphi(t - r_i(t))}{\sum_{j=1}^{n} \alpha_j(t)\varphi(t - r_j(t))}.$$

So both equations (4) and (6) are equivalent in this case.

3. Applications—convergence criteria and sufficient convergence conditions

In this section we develop several sufficient conditions for convergence of all solutions of some of the classes of equations considered in Section 2. Let us note that it is possible to develop appropriate sufficient convergence criteria for all the remaining classes of equations indicated in this section.

3.1. Linear case

The equation of the type (4) with $n = 1$, $\beta \in C(I, \mathbb{R}^+)$, $\beta_1 \equiv 1$ and $r_1(t) \equiv r =$ const, namely, the equation

$$\dot{x}(t) = \beta(t)\bigl[x(t) - x(t - r)\bigr] \tag{7}$$

was widely considered under various suppositions (see, e.g., [2–4,7–11,14,20, 21]). Some similar classes of equations are studied in the papers [1,5,6] and [16] as well. The case of convergence of all its solutions, as well as the case of existence of a divergent solution, were discussed.

Although Eq. (7) was considered in the papers mentioned, it is possible to give, with the aid of Theorem 4, not only generalizations of the known criteria, but the new criteria also. The next corollary is a consequence of Theorem 4 and the inequality (3) with $\lambda(t) \equiv \beta(t)\lambda^*(t)$ and $f(t, x_t) \equiv \beta(t)[x(t) - x(t - r)]$.

Corollary 2. *A necessary and sufficient condition for the convergence of all solutions of* (7) *corresponding to the initial point t_0 is the existence of a function $\lambda^* \in C(I, \mathbb{R}^+)$ satisfying the integral inequality*

$$1 + \lambda^*(t) \geqslant \exp\left[\int_{t-r}^{t} \beta(s)\lambda^*(s)\, \mathrm{d}s\right] \tag{8}$$

on interval I_0.

This criterion can serve as a source of new very sharp sufficient conditions. We will formulate some of them. Before that, we define necessary auxiliary notions.

Definition 1. Let us define the expression $\ln_k t$, $k \geqslant 1$, by the formula $\ln_k t = \ln(\ln_{k-1} t)$, $\ln_0 t \equiv t$ where $t > \exp_{k-2} 1$ and $\exp_s t = \exp(\exp_{s-1} t)$, $s \geqslant 1$,

$\exp_0 t \equiv t$ and $\exp_{-1} t \equiv 0$ (instead of the expressions $\ln_0 t$, $\ln_1 t$, we write only t and $\ln t$ in the following).

Introduce, for $t > \exp_{n-2} 1$, auxiliary functions

$$L_n^p(t) = 1 - \frac{r}{t} - \frac{r}{t\ln t} - \cdots - \frac{r}{t\ln t \ldots \ln_{n-1} t} - \frac{pr}{t\ln t \ldots \ln_n t}$$

where $n \geqslant 0$ and p is a constant and

$$\lambda_n^\varepsilon(t) = \frac{1}{t\ln t \ldots \ln_{n-1} t \ln_n^\varepsilon t}$$

where $n \geqslant 0$ and ε is a constant.

In the following, we will suppose the independent variable t sufficiently large so that all the functions used are well defined.

3.1.1. The point test of convergence

Theorem 5. *Let for all $t \in I$ with sufficiently large t_0, fixed $n \in \mathbb{N} \cup \{0\}$ and a constant $p > 1$:*

$$\beta(t) \leqslant B_n^p(t) \equiv \frac{1}{2r} + \frac{1}{2r} L_n^p(t). \tag{9}$$

Then each solution of Eq. (7) *corresponding to the initial point t_0 converges.*

Example 1. Let us illustrate the sharpness of Theorem 5. Consider an equation of the type (7) with $\beta(t) \equiv B_n^a(t)$:

$$\dot{x}(t) = B_n^a(t)\bigl[x(t) - x(t-r)\bigr], \quad n \in \mathbb{N} \cup \{0\}. \tag{10}$$

The inequality (9) holds and by Theorem 5 each solution of (10) *converges* if $a \geqslant p > 1$. *Nevertheless,* for $a = p = 1$ this is already not true. Let us show this: By [9, Theorem 2] equation (7) has a solution $x = X(t)$ with property $X(+\infty) = +\infty$ if and only if the differential inequality

$$\dot{\omega}(t) \leqslant \beta(t)\bigl[\omega(t) - \omega(t-r)\bigr] \tag{11}$$

has a solution with the same property. It is easy to see that a solution of inequality (11) with $\beta(t) \equiv B_n^1(t)$ has the form $\omega(t) = \ln_{n+1} t$ and $\omega(+\infty) = +\infty$. Really, in this case $\dot{\omega}(t) = (t\ln t \ldots \ln_n t)^{-1}$ and (with the aid of Lemma 1 (Section 4.1) for $\sigma = 1$, $k = n+1$)

$$\begin{aligned} B_n^1(t)\bigl[\ln_{n+1} t - \ln_{n+1}(t-r)\bigr] &= \left(\frac{1}{r} - \frac{1}{2t} - \cdots - \frac{1}{2t\ln t \ldots \ln_n t}\right) \\ &\quad \times \left(\frac{r}{t\ln t \ldots \ln_n t} + \frac{r^2}{2t^2 \ln t \ldots \ln_n t} + \cdots + \frac{r^2}{2(t\ln t \ldots \ln_n t)^2}\right. \\ &\quad \left. + \frac{r^3(1+o(1))}{3t^3 \ln t \ldots \ln_n t}\right) \end{aligned}$$

$$= \frac{1}{t \ln t \ldots \ln_n t} + \frac{r^2(1+o(1))}{12t^3 \ln t \ldots \ln_n t}.$$

Now inequality (11) holds for $t \to +\infty$.

So, for $a = 1$ the property of convergence of all solutions of Eq. (10) does not hold.

3.1.2. *The integral test of convergence*

Observe that in Theorem 5, the conditions are formulated in terms of the values of the function β itself (type A) and not in terms of the average $r^{-1} \cdot \int_t^{t+r} \beta(s)\,\mathrm{d}s$ or a weighted average $\int_t^{t+r} \beta(s)\varphi(s)\,\mathrm{d}s$ (type B) with a suitable function φ. Sometimes conditions of the type A can yield stronger results. In general, however, conditions of the type B are preferable. In the next theorem, we establish the conditions of the type B.

Theorem 6. *Let for all $t \in I$ with sufficiently large t_0, fixed $n \in \mathbb{N} \cup \{0\}$, and a constant $p > 1$ there exists a function $\varphi \in C(I, \mathbb{R}^+)$ such that the inequality*

$$\int_t^{t+r} \beta(s)\varphi(s)\,\mathrm{d}s \leqslant \varphi(t) L_n^p(t) \tag{12}$$

holds. If, moreover, the ratio $\lambda_n^\varepsilon(t)/\varphi(t)$ is a function nonincreasing on I for every fixed constant ε within an interval $(1, 1+\varepsilon_0)$ with $\varepsilon_0 > 0$, $\varepsilon_0 = \mathrm{const}$, then each solution of Eq. (7) *corresponding to the initial point t_0 converges.*

Example 2. Let us now demonstrate the sharpness of Theorem 6. Consider the previous equation (10) and put

$$\varphi(t) = \frac{1}{t \ln t \ldots \ln_n t}.$$

The ratio $\lambda_n^\varepsilon(t)/\varphi(t) \equiv 1/\ln_n^{\varepsilon-1} t$ is a nonincreasing function for $t \to +\infty$ and for every fixed $\varepsilon > 1$. The right-hand side of inequality (12) equals

$$\begin{aligned}\mathcal{R} &= \varphi(t) L_n^p(t) \\ &\equiv \frac{1}{t \ln t \ldots \ln_n t}\left[1 - \frac{r}{t} - \frac{r}{t \ln t} - \cdots - \frac{r}{t \ln t \ldots \ln_{n-1} t} - \frac{pr}{t \ln t \ldots \ln_n t}\right].\end{aligned}$$

The left-hand side of the inequality (12) equals

$$\begin{aligned}\mathcal{L} &\equiv \int_t^{t+r} \beta(s)\varphi(s)\,\mathrm{d}s = \int_t^{t+r} B_n^a(s)\varphi(s)\,\mathrm{d}s \\ &= \int_t^{t+r} \left[\frac{1}{r} - \frac{1}{2s} - \frac{1}{2s \ln s} - \cdots - \frac{1}{2s \ln s \ldots \ln_{n-1} s} - \frac{a}{2s \ln s \ldots \ln_n s}\right]\end{aligned}$$

$$\times \frac{1}{s \ln s \dots \ln_n s}\, \mathrm{d}s$$
$$= \frac{1}{r} \cdot \mathcal{J}_0 - \frac{1}{2} \cdot \sum_{i=1}^{n} \mathcal{I}_i - \frac{a}{2} \cdot \mathcal{I}_{n+1}$$

with

$$\mathcal{J}_0 = \int_t^{t+r} \frac{\mathrm{d}s}{s \ln s \dots \ln_n s} \quad \text{and} \quad \mathcal{I}_i = \int_t^{t+r} \frac{\mathrm{d}s}{(s \ln s \dots \ln_i s)^2 \ln_{i+1} s \cdots \ln_n s}.$$

The first integral $\mathcal{J}_0$ admits the following asymptotic representation:

$$\mathcal{J}_0 = \ln_{n+1} s\big|_t^{t+r} = \ln \frac{\ln_n(t+r)}{\ln_n t}$$
$$= \ln\left[1 + \frac{r}{t \ln t \dots \ln_n t} - \frac{r^2}{2t^2 \ln t \dots \ln_n t} - \frac{r^2}{2(t \ln t)^2 \ln_2 t \dots \ln_n t} - \cdots \right.$$
$$\left. - \frac{r^2}{2(t \ln t \dots \ln_{n-1} t)^2 \ln_n t} + o\left(\frac{1}{t^3}\right)\right]$$

(by Lemma 1 (Section 4.1) with $r \sim -r$ and $\sigma = 1$)

$$= \frac{r}{t \ln t \dots \ln_n t} - \frac{r^2}{2t^2 \ln t \dots \ln_n t} - \frac{r^2}{2(t \ln t)^2 \ln_2 t \dots \ln_n t} - \cdots$$
$$- \frac{r^2}{2(t \ln t \dots \ln_{n-1} t)^2 \ln_n t} - \frac{r^2}{2(t \ln t \dots \ln_n t)^2} + o\left(\frac{1}{t^3}\right)$$

(by Maclaurin's formula).

Now we estimate ($i = 1, \dots, n+1$)

$$\mathcal{I}_i > \frac{1}{(t+r) \ln(t+r) \dots \ln_i (t+r)} \int_t^{t+r} \frac{\mathrm{d}s}{s \ln s \dots \ln_n s}$$
$$= \frac{1}{(t+r) \ln(t+r) \dots \ln_i (t+r)} \cdot \mathcal{J}_0$$
$$= \frac{r}{(t \ln t \dots \ln_i t)^2 \ln_{i+1} t \dots \ln_n t} + o\left(\frac{1}{t^3}\right).$$

Connecting the above computations, we are able to estimate $\mathcal{L}$:

$$\mathcal{L} < \frac{1}{t \ln t \dots \ln_n t} - \frac{r}{t^2 \ln t \dots \ln_n t} - \frac{r}{(t \ln t)^2 \ln_2 t \dots \ln_n t} - \cdots$$
$$- \frac{r}{(t \ln t \dots \ln_{n-1} t)^2 \ln_n t} - \frac{r(1+a)}{2(t \ln t \dots \ln_n t)^2} + o\left(\frac{1}{t^3}\right)$$

$$= \frac{1}{t\ln t \dots \ln_n t}$$
$$\times \left[1 - \frac{r}{t} - \frac{r}{t\ln t} - \dots - \frac{r}{t\ln t \dots \ln_{n-1} t} - \frac{(1+a)r}{2t\ln t \dots \ln_n t} + o\left(\frac{1}{t^3}\right)\right].$$

Comparing the corresponding terms, we conclude that for $\mathcal{L} < \mathcal{R}$ the inequality

$$-\frac{r}{2}(1+a) < -pr$$

is a sufficient condition. This inequality turns into $a > 2p - 1$. Since this inequality guarantees the validity of inequality (12) and we can take $p\,(> 1)$ sufficiently close to number 1, we conclude that convergence of all solutions will be guaranteed if $a > 1$. Thus with the aid of the integral criterion we get, by Theorem 6, a sharp result again.

Remark 3. Let us note that for $n = 1$, the condition (3.10) of paper [4] is a consequence of Theorem 5. Moreover, a partial case of Theorem 6 for $n = 1$ and $\varphi \equiv 1$ is a criterion given in [4, Theorem 3.3].

3.2. Nonlinear case

Let us consider the equation of the type (5) with $\tau(t) \equiv r = \text{const}$, $\alpha \in (0, 1)$, $\alpha = \text{const}$, $\beta \in C(I, \mathbb{R}^+)$ and $\gamma, \delta \in C(I_0, \mathbb{R}^+)$, i.e. the equation

$$\dot{x}(t) = \frac{\beta(t)[x(t) - x(t-r)]}{\gamma(t) + \delta(t)|x(t) - x(t-r)|^\alpha}, \quad t \in I_0. \tag{13}$$

(The case $\alpha = 0$ was considered in Section 3.1.) According to Theorem 4, all solutions of Eq. (13) converge if and only if there exist a positive constant m and a function $\lambda : I \to \mathbb{R}^+$ such that for $t \in I_0$ inequality (3) holds if

$$f(t, x_t) \equiv \frac{\beta(t)[x(t) - x(t-r)]}{\gamma(t) + \delta(t)|x(t) - x(t-r)|^\alpha}.$$

The next corollary is a consequence of this theorem with $\lambda(t) \equiv \beta(t)L(t)$.

Corollary 3. *For the convergence of all solutions of* (13) *corresponding to the initial point t_0, the existence of a function $L \in C(I, \mathbb{R}^+)$ and a positive constant m, satisfying the integral inequality*

$$L(t)\left[\gamma(t) + \delta(t)m^\alpha \mathrm{e}^{-\alpha\int_{t_0}^t \beta(s)L(s)\,\mathrm{d}s} \cdot \left(\mathrm{e}^{\int_{t-r}^t \beta(s)L(s)\,\mathrm{d}s} - 1\right)^\alpha\right]$$
$$\geqslant \mathrm{e}^{\int_{t-r}^t \beta(s)L(s)\,\mathrm{d}s} - 1 \tag{14}$$

on interval I_0 is both necessary and sufficient.

Let us give sufficient conditions for the convergence of all solutions of Eq. (13).

Theorem 7. *Suppose that $\gamma(t) \equiv \delta(t) \equiv 1$ and $\beta(t) \leqslant C/t^q$ for all $t \in I$ with sufficiently large t_0, some fixed positive constant C and a constant $q > \alpha$. Then each solution of Eq.* (13) *corresponding to the initial point t_0 converges.*

4. Auxiliary results and proofs

4.1. Asymptotic expansion of iterated logarithms

The following lemma (which is used in this paper several times) can be proved in an elementary way by the method of induction. Therefore its proof is omitted.

Lemma 1. *For fixed $r, \delta \in \mathbb{R} \setminus \{0\}$, $k \geqslant 1$ and for $t \to \infty$ the following asymptotic representation holds:*

$$\begin{aligned}\frac{\ln_k^\delta(t-r)}{\ln_k^\delta t} = 1 &- \frac{r\delta}{t\ln t \ldots \ln_k t} - \frac{r^2\delta}{2t^2\ln t\ldots\ln_k t} - \frac{r^2\delta}{2(t\ln t)^2\ln_2 t\ldots\ln_k t}\\ &- \cdots - \frac{r^2\delta}{2(t\ln t\ldots\ln_{k-1}t)^2\ln_k t} + \frac{r^2\delta(\delta-1)}{2(t\ln t\ldots\ln_k t)^2}\\ &- \frac{r^3\delta(1+o(1))}{3t^3\ln t\ldots\ln_k t}.\end{aligned}$$

4.2. Topological principle

Consider the system of functional differential equations of retarded type

$$\dot{y}(t) = g(t, y_t) \tag{15}$$

where $y \in \mathbb{R}^n$, y_t is an element of the space of continuous functions $\mathcal{C}_n = C_n([-r,0],\mathbb{R}^n)$, $y_t(\theta) = y(t+\theta)$ where $\theta \in [-r, 0]$, $g : \Omega^* \mapsto \mathbb{R}^n$, Ω^* is an open subset of $\mathbb{R} \times \mathcal{C}_n$ and g is a continuous mapping such that the element $(\delta, \varphi) \in \Omega^*$ determines a unique solution $y(\delta, \varphi)$ on its maximal existence interval.

In the proof of Theorem 3, which is crucial in this paper, the topological principle of Ważewski [19] in the modified form given by Rybakowski [18] (for retarded functional differential equations) is used. This modification applies Razumikhin's approach concerning the estimation of the full derivative of Liapunov function along the trajectories of considered system. Let us give a summary of this principle.

As usual, if a set $\omega \subset \mathbb{R} \times \mathbb{R}^n$, then $\operatorname{int}\omega$, $\overline{\omega}$ and $\partial\omega$ denote the interior, the closure and the boundary of ω, respectively.

4.2.1. Retract

The main role of the topological principle play notions of retract and retraction.

Definition 2 [15, p. 97]. If $\mathcal{A} \subset \mathcal{B}$ are any two sets of a topological space and $\pi : \mathcal{B} \to \mathcal{A}$ is a continuous mapping from $\mathcal{B}$ onto $\mathcal{A}$ such that $\pi(p) = p$ for every $p \in \mathcal{A}$, then π is called a *retraction* of $\mathcal{B}$ onto $\mathcal{A}$. When there exists a retraction of $\mathcal{B}$ onto $\mathcal{A}$, $\mathcal{A}$ is called a *retract* of $\mathcal{B}$.

4.2.2. Regular polyfacial set

Applications of the topological principle are mostly made by using a construction of the so-called regular polyfacial set ω, the definition of which is given below. The meaning of this construction, in the case of ordinary differential equations, consists in the fact that each egress point of the boundary $\partial\omega$ is at the same time the point of strict egress (for corresponding definitions of points of egress and strict egress we refer, e.g., to the books [13,15]). This property is verified, as a rule, by means of computing the full derivatives (along the trajectories of a given system of differential equations) of functions that generate the boundary $\partial\omega$, and by means of estimating their signs.

In the case of retarded functional differential equations, the regular polyfacial set is constructed in the same way. Nevertheless, the signs of the full derivatives (along the trajectories of a given system of retarded functional differential equations) of functions whose graphs generate the boundary $\partial\omega$ are, in view of the delay phenomena, estimated in accordance with Razumikhin's idea (e.g., [17]). To determine if a point of the boundary $\partial\omega$ could serve as an analogy of egress (strict egress) point, etc., it is supposed (following this idea) that the corresponding part of graph-history (which is used in computations) of the considered part of the solution lies in $\operatorname{int}\omega$.

Definition 3. Let the functions $l_i(t, y)$, $i = 1, \dots, p$, and $m_j(t, y)$, $j = 1, \dots, q$, $p + q > 0$ be continuous on some set $\omega_0 = \omega_0^1 \times \omega_0^2 \subset \mathbb{R} \times \mathbb{R}^n$ where the set ω_0^2 is open. (If $p = 0$ or $q = 0$ then corresponding functions are not considered.) Moreover, we will suppose continuous differentiability of these functions at the points indicated below. The set

$$\omega = \left\{(t, y) \in \omega_0\colon\ l_i(t, y) < 0,\ m_j(t, y) < 0,\ i = 1, \dots, p,\ j = 1, \dots, q\right\}$$

is called a *regular polyfacial set* with respect to the system (15) if (α) through (γ) hold:

(α) For $(t, \pi) \in \mathbb{R} \times \mathcal{C}_n$ such that $(t + \theta, \pi(\theta)) \in \omega$ for $\theta \in [-r, 0)$, we have $(t, \pi) \in \Omega^*$.

(β) For all $i = 1, 2, \dots, p$, all $(t, y) \in \partial\omega$ for which $l_i(t, y) = 0$, and all $\pi \in \mathcal{C}_n$ for which $\pi(0) = y$ and $(t + \theta, \pi(\theta)) \in \omega$, $\theta \in [-r, 0)$ it follows that $Dl_i > 0$ where

$$Dl_i(t, y) \equiv \sum_{s=1}^{n} \frac{\partial l_i(t, y)}{\partial y_s} \cdot g_s(t, \pi) + \frac{\partial l_i(t, y)}{\partial t}.$$

(γ) For all $j = 1, 2, \dots, q$, all $(t, y) \in \partial\omega$ for which $m_j(t, y) = 0$, and all $\pi \in \mathcal{C}_n$ for which $\pi(0) = y$ and $(t + \theta, \pi(\theta)) \in \omega$, $\theta \in [-r, 0)$ it follows that $Dm_j < 0$ where

$$Dm_j(t, y) \equiv \sum_{s=1}^{n} \frac{\partial m_j(t, y)}{\partial y_s} \cdot g_s(t, \pi) + \frac{\partial m_j(t, y)}{\partial t}.$$

In the following, the elements $(t, \pi) \in \mathbb{R} \times \mathcal{C}_n$ are assumed to be such that $(t, \pi) \in \Omega^*$.

4.2.3. System of initial functions

Successive application of Razumikhin approach leads to the necessity of involving the notion of system of the so-called *initial functions*. Each of these functions is defined on an interval of the length r and the following definition describes their properties. In the proof of Theorem 3, a special system of initial functions will be used. The advantage of using the topological principle consists in the fact that the set of initial functions can be prescribed. In the proofs of Theorems 3 and 4, such possibility is used. Note that using a stationary point principle can cause difficulties since each constant is a solution of Eq. (1). So, it is not clear how to construct an appropriate space of functions which could serve as the domain of definition for the corresponding operator.

Definition 4. A *system of initial functions* $p_{A,\omega}$ with respect to the nonempty sets A and ω where $A \subset \overline{\omega} \subset \mathbb{R} \times \mathbb{R}^n$ is defined as a continuous mapping $p : A \to \mathcal{C}_n$ such that (α) and (β) below hold:

(α) If $z = (t, y) \in A \cap \operatorname{int} \omega$, then $(t + \theta, p(z)(\theta)) \in \omega$ for $\theta \in [-r, 0]$.

(β) If $z = (t, y) \in A \cap \partial\omega$, then $(t + \theta, p(z)(\theta)) \in \omega$ for $\theta \in [-r, 0)$ and $(t, p(z)(0)) = z$.

4.2.4. Existence theorem

The following theorem, concerning the existence of solution of system (15) with graph remaining in the set ω on its maximal interval existence, describes the main result of the paper [18]. (A geometrical illustration is given in Fig. 1.)

Theorem 8. *Let ω be a regular polyfacial set with respect to the system* (15) *and let W be defined as follows*:

$$W = \big\{(t, y) \in \partial\omega\colon\ m_j(t, y) < 0,\ j = 1, 2, \dots, q\big\}.$$

Let $Z \subset W \cup \omega$ be a given set such that $Z \cap W$ is a retract of W but not a retract of Z. Then for each fixed system of initial functions $p_{Z,\omega}$ there exists a point $z_0 = (\sigma_0, y_0) \in Z \cap \omega$ such that for the corresponding solution $y(\sigma_0, p(z_0))$ of (15) *we have $(t, y(\sigma_0, p(z_0))(t)) \in \omega$ for each $t \in I_{\sigma_0, p(z_0)}$.*

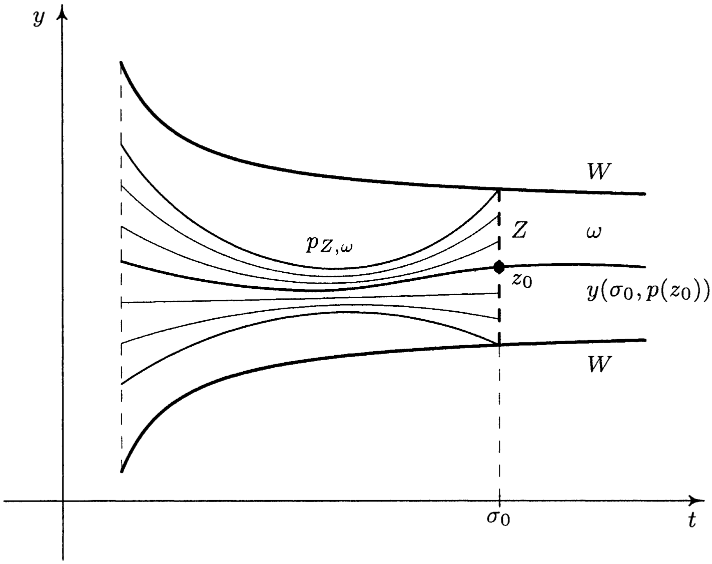

Fig. 1. Illustration to Theorem 8.

4.3. *Proof of Theorem 1*

Solution $x(t_0, \varphi_{t_0})$ with $(t_0, \varphi_{t_0}) \in \Omega$ and with increasing initial function $\varphi(t_0 + \theta)$ for $\theta \in [-r, 0]$ generates in view of (v) and (β) a solution of Eq. (1) with $I_{t_0,\varphi_{t_0}} = [t_0, \infty)$ which is increasing and convergent. Consequently, (α) holds. The same is true for every $x(\sigma, \varphi_\sigma)$ with $\sigma \geqslant t_0$. Function $-x(t_0, \varphi_{t_0})$ is in view of (ii) a decreasing and convergent solution of Eq. (1) and the function $\nu \equiv -x(t_0, \varphi_{t_0})$ turns the inequality (2) into an equality. Therefore (γ) holds too. The theorem is proved.

4.4. *Proof of Theorem 2*

4.4.1. *The case* $(\gamma) \Rightarrow (\delta)$

Without loss of generality, we suppose that the function ν is positive on I_0. If not, then we generate a new function ν^* with the same properties as the previous one, by means of an appropriate shift by a constant. This is possible in view of condition (i). Since the function ν is decreasing, it is easy to see that a positive function $\lambda : I \to \mathbb{R}^+$ satisfying the relation

$$\nu(t) = \nu(t_0) e^{-\int_{t_0}^{t} \lambda(s)\, ds} \tag{16}$$

exists. Then the inequality (3) with $m = \nu(t_0)$ immediately follows from (2).

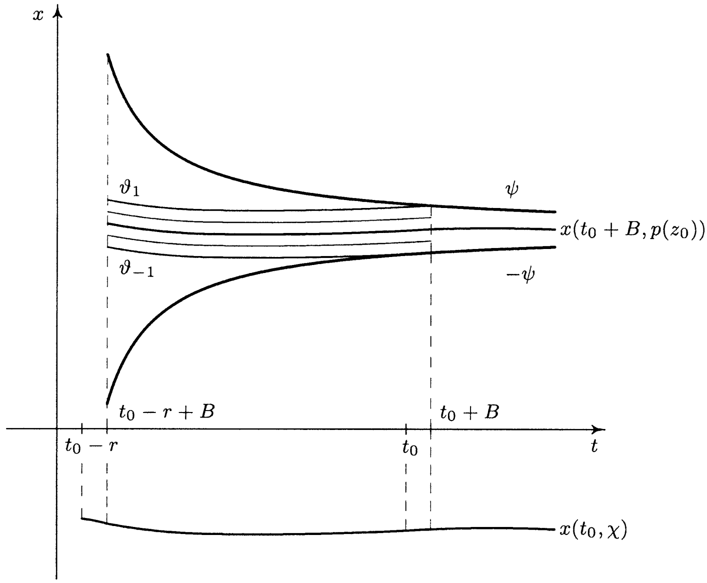

Fig. 2. Special system of initial functions.

4.4.2. The case $(\delta) \Rightarrow (\gamma)$

The inequality (2) follows from inequality (3) in which, in accordance with (16), we put

$$\lambda(t) = -\frac{\dot{\nu}(t)}{\nu(t)}, \qquad \nu(t_0) = m.$$

The theorem is proved.

4.5. Proof of Theorem 3

We will follow the following scheme of the proof: at first let us prove $(\alpha) \Rightarrow (\beta)$. Then we show that $(\alpha) \Rightarrow (\gamma)$. This will end the proof since the remaining part of the proof follows from Theorem 2. Some parts of the proof are accompanied by Fig. 2. (In this figure the functions ψ (see formula (17) below) and $-\psi$ are (without loss of generality) shifted by a constant in order to make this figure more visual.)

4.5.1. The case $(\alpha) \Rightarrow (\beta)$

(a) Constructing the regular polyfacial set with respect to Eq. (1)

By (α), (i) and (ii) there exists a decreasing positive solution $x(t) = \Phi(t)$ of Eq. (1) on I which tends to zero, i.e., $\Phi(\infty) = 0$. Without loss of generality, we prove the property (β) on interval I_0 instead of the general interval $[\sigma, \infty)$

with $\sigma \geqslant t_0$. This possibility follows from the method of the proof, which can be modified for any interval $[\sigma, \infty)$ with $\sigma \geqslant t_0$ easily. Let us define an auxiliary function

$$\psi(t) \equiv \delta\Phi(t) \tag{17}$$

where δ is a positive constant, $\delta \geqslant k_0$. A more detailed choice of this constant will be given in Section 4.5.1(b). Define the set

$$\omega_0^* \equiv \{(t, x)\colon\ t \in [B^0, \infty),\ x \in \mathbb{R}\} \quad \text{with } B^0 \in (t_0 - r, t_0)$$

and put

$$\omega^* = \{(t, x) \in \omega_0^*\colon\ l_1(t, x) < 0\}$$

with

$$l_1(t, x) \equiv \big(x - \psi(t)\big)\big(x + \psi(t)\big).$$

We show that the set ω^* is a regular polyfacial set with respect to Eq. (1). For this we put $n = 1$, $p = 1$, $q = 0$, $\mathcal{C}_1 \equiv \mathcal{C}$, $\Omega = \Omega^*$, $\omega \equiv \omega^*$ and $\omega_0 \equiv \omega_0^*$ in Definition 3. The condition (α) of Definition 3 is obviously fulfilled. Let us verify the condition (β) of Definition 3. For $t > B^0 + r$ (this interval is taken in accordance with properties of function π—i.e., $(t+\theta, \pi(\theta)) \in \omega^*$ if $\theta \in [-r, 0)$—involved in the condition (β) of Definition 3) we get:

$$\begin{aligned} Dl_1(t, x) = {} & \big(f(t, x_t) - \delta f(t, \Phi_t)\big)\big(x + \psi(t)\big) \\ & + \big(x - \psi(t)\big)\big(f(t, x_t) + \delta f(t, \Phi_t)\big). \end{aligned}$$

This derivative is computed under the assumption that the end point of the function π (see Definition 3) reaches the boundary $\partial\omega^*$. In our case the boundary points are given by the relation $(t, x) \in \partial\omega^*$, i.e., $x = \pm\psi(t)$. Let us consider both cases separately: If $x = \psi(t)$, then in view of properties (iii), (iv):

$$\begin{aligned} Dl_1|_{x=\psi(t)} &= 2\psi(t) \cdot \big(f(t, x_t) - \delta f(t, \Phi_t)\big)\big|_{x(t)=\psi(t)} \\ &> 2\psi(t) \cdot \big(f(t, \delta\Phi_t) - \delta f(t, \Phi_t)\big) \quad \text{(by (iv), } \varphi_t \equiv \psi_t) \\ &\geqslant 2\psi(t) \cdot \big(\delta f(t, \Phi_t) - \delta f(t, \Phi_t)\big) = 0 \quad \text{(by (iii), } x_t \equiv \Phi_t). \end{aligned}$$

Thus $Dl_1|_{x=\psi(t)} > 0$. Moreover, if $x = -\psi(t)$, then in view of properties (ii), (iii) and (iv):

$$\begin{aligned} Dl_1|_{x=-\psi(t)} &= -2\psi(t) \cdot \big(f(t, x_t) + \delta f(t, \Phi_t)\big)\big|_{x(t)=-\psi(t)} \\ &= 2\psi(t) \cdot \big(f(t, -x_t) - \delta f(t, \Phi_t)\big)\big|_{-x(t)=\psi(t)} \quad \text{(by (ii))} \\ &= 2\psi(t) \cdot \big(f(t, v_t) - \delta f(t, \Phi_t)\big)\big|_{v(t)=\psi(t)} \quad \text{(substitute } x \equiv -v) \\ &> 2\psi(t) \cdot \big(f(t, \delta\Phi_t) - \delta f(t, \Phi_t)\big) \quad \text{(by (iv), } \varphi_t \equiv \psi_t) \\ &\geqslant 2\psi(t) \cdot \big(\delta f(t, \Phi_t) - \delta f(t, \Phi_t)\big) = 0 \quad \text{(by (iii), } x_t \equiv \Phi_t) \end{aligned}$$

and $Dl_1|_{x=-\psi(t)} > 0$. So, under the suppositions described in condition (β) of Definition 3: $Dl_1|_{(t,x)\in\partial\omega^*} > 0$. Condition (β) is satisfied. Condition (γ) of Definition 3 is not used. Consequently, the set ω^* is the regular polyfacial set with respect to Eq. (1).

(b) Construction of auxiliary functions ϑ_1, ϑ_{-1} and choice of the constant δ

Let $x(t) = x(t_0, \chi)(t)$ be the solution of Eq. (1) (passing through $(t_0, \chi) \in \Omega$) generated by a nonconstant initial function χ and defined on an interval $[t_0 - r, t_0 + A)$ with $A > 0$ (see Fig. 2). Let $0 < B < \min\{A, r\}$. Furthermore, without loss of generality, let us suppose $t_0 + B > B^0 + r$.

One of the crucial moments of the proof is the requirement that for sufficiently large δ the graphs of the functions (which are generated by indicated shifts of the solution $x(t)$ and are itself solutions of Eq. (1)):

$$\vartheta_1(t) \equiv x(t_0, \chi)(t) - x(t_0, \chi)(t_0 + B) + \psi(t_0 + B),$$
$$\vartheta_{-1}(t) \equiv x(t_0, \chi)(t) - x(t_0, \chi)(t_0 + B) - \psi(t_0 + B)$$

lie in ω^* on interval $[t_0 + B - r, t_0 + B)$. Below we show that this is possible. This property will permit us to construct a system of initial functions with certain prescribed properties.

Denote

$$M_x = \max_{[t_0, t_0+B]} \left|\dot{x}(t_0, \chi)(t)\right|, \qquad m_\Phi = \min_{[t_0, t_0+B]} \left|\dot{\Phi}(t)\right| \neq 0,$$

and

$$M_E = \max_{(v,s)\in[t_0-r+B, t_0]} \left|x(v) - x(s)\right|.$$

Clearly, $\dot{\psi}(t) = \delta\dot{\Phi}(t)$. Then $\min_{[t_0, t_0+B]} \dot{\psi}(t) = \delta m_\Phi$ and for validity of the inequality

$$\min_{[t_0, t_0+B]} \dot{\psi}(t) \geqslant M_x,$$

it is sufficient $\delta \geqslant M_x m_\Phi^{-1}$. Moreover, for the validity of the inequality $\psi(t_0) - \vartheta_1(t_0) \geqslant M_E$ the inequality

$$\delta \geqslant \frac{M_E + x(t_0, \chi)(t_0) - x(t_0, \chi)(t_0 + B)}{\Phi(t_0) - \Phi(t_0 + B)}$$

is sufficient and for the validity of the inequality $\vartheta_{-1}(t_0) + \psi(t_0) \geqslant M_E$ the inequality

$$\delta \geqslant \frac{M_E - x(t_0, \chi)(t_0) + x(t_0, \chi)(t_0 + B)}{\Phi(t_0) - \Phi(t_0 + B)}$$

is sufficient. In the following we suppose

$$\delta > \delta_0 := \max\left\{k_0, \frac{M_x}{m_\Phi}, \frac{M_E \pm [x(t_0, \chi)(t_0) - x(t_0, \chi)(t_0 + B)]}{\Phi(t_0) - \Phi(t_0 + B)}\right\}. \tag{18}$$

By our construction $\vartheta_1(t_0+B)=\psi(t_0+B)$ and

$$\vartheta_1(t)<\psi(t), \quad t\in[t_0-r+B,t_0+B). \tag{19}$$

Indeed, in view of the inequality

$$\left|\dot{\vartheta}_1(t)\right|=\left|\dot{x}(t_0,\chi)(t)\right|<\dot{\psi}(t), \quad t\in[t_0,t_0+B),$$

the inequality (19) holds on the interval $[t_0,t_0+B)$ and in view of the inequality

$$\begin{aligned}&\max_{(v,s)\in[t_0-r+B,t_0]}\left|\vartheta_1(v)-\vartheta_1(s)\right|\\ &\quad=\max_{(v,s)\in[t_0-r+B,t_0]}\left|x(v)-x(s)\right|=M_E<\psi(t_0)-\vartheta_1(t_0)\end{aligned}$$

the inequality (19) holds on the interval $[t_0-r+B,t_0]$. Similarly it can be verified that $\vartheta_{-1}(t_0+B)=-\psi(t_0+B)$ and

$$\vartheta_{-1}(t)>-\psi(t), \quad t\in[t_0-r+B,t_0+B). \tag{20}$$

(c) Special system of initial functions

In this part of the proof, a special system of initial functions will be constructed. The main property of this family of functions is that the distance between any two functions is equal to a *constant* and that each function from this family is *bounded* (in absolute value) by auxiliary function ψ given by (17) with a parameter $\delta>\delta_0$ (see (18)). This construction uses solution $x(t_0,\chi)$ (see Fig. 2). This system of initial functions will consist of functions generated by appropriate shifts of a part of the given solution $x(t_0,\chi)$. Such a special system will permit us to end this part of the proof.

Let us construct a family of curves $\vartheta_\mu(t)$ with $\mu\in[-1,1]$, $t\in[t_0-r+B, t_0+B]$ as

$$\vartheta_\mu(t)\equiv x(t_0,\chi)(t)-x(t_0,\chi)(t_0+B)+\mu\psi(t_0+B).$$

It is easy to see that $\vartheta_{\pm1}(t_0+B)=\pm\psi(t_0+B)$ and $\vartheta_\mu(t_0+B)\in(\vartheta_{-1}(t_0+B), \vartheta_1(t_0+B))$ for any $\mu\in(-1,1)$. Also in view of (18), inequalities (19), (20) hold, as was shown in the previous part (b).

Let us summarize: for each $\mu\in[-1,1]$,

$$|\vartheta_\mu(t)|<\psi(t), \quad t\in[t_0-r+B,t_0+B),$$

for each $\mu\in(-1,1)$:

$$\left|\vartheta_\mu(t_0+B)\right|<\psi(t_0+B),$$

and $\vartheta_{\pm1}(t_0+B)=\pm\psi(t_0+B)$. This means, in accordance with Definition 4, that the set of functions p_{A,ω^*} defined for the set

$$A=\left\{(t,x)\in\overline{\omega}^*,\ t=t_0+B,\ -\psi(t_0+B)\leqslant x\leqslant\psi(t_0+B)\right\}$$

as mapping

$$p(z)(t)=\vartheta_{\mu^*}(t)\to\mathcal{C}$$

where $z = (t_0 + B, x^*) \in A$ and $\mu^* = x^*/\psi(t_0 + B)$ generates the system of initial functions with respect to A and ω^*. Clearly, if x^* varies over the whole interval $[-\psi(t_0 + B), \psi(t_0 + B)]$, μ^* varies over the whole interval $[-1, 1]$.

(d) Convergence of solution $x(t_0, \chi)$ and convergence of all solutions

Let us put $W = \{(t, x) \in \partial\omega^*\}$ and $Z \equiv A$. It is easy to see that $Z \cap W$ is a retract of W in view of continuous mapping

$$W \ni (t, x) \mapsto (t_0 + B, \tilde{x}) \in Z \cap W$$

with $\tilde{x} = x\psi(t_0 + B)/\psi(t)$. Points of the set $Z \cap W$ are fixed for this mapping and conditions of Definition 2 are met for $\mathcal{A} \equiv \mathcal{Z} \cap \mathcal{W}$ and $\mathcal{B} \equiv W$. Taking the system of initial functions p_{A,ω^*} defined in part (c), we see that all the assumptions of Theorem 8 hold. Then there exists a point $z_0 = (t_0 + B, x_0) \in Z \cap \omega^*$ such that for the corresponding solution $x(t_0 + B, p(z_0))$ of Eq. (1) we have: $(t, x(t_0 + B, p(z_0))(t)) \in \omega^*$ for each $t \in D_{t_0+B,p(z_0)}$.

Let us prove that $D_{t_0+B,p(z_0)} \equiv [t_0 + B, \infty)$. In view of the construction, the inequality

$$\left|x\left(t_0 + B, p(z_0)(t)\right)\right| < \psi(t) \tag{21}$$

is valid on $D_{t_0+B,p(z_0)}$. Let us suppose, on the contrary, that $D_{t_0+B,p(z_0)} \equiv [t_0 + B, t_0 + B + \varepsilon)$ with a positive ε. Due to the conditions imposed on f and in view of (21), it is possible to define this solution on $[t_0 + B, t_0 + B + \varepsilon]$ and then to continue it on an interval $[t_0 + B, t_0 + B + \varepsilon_1)$ with $\varepsilon_1 > \varepsilon$ and so on. Due to inequality (21), the continuability is proved for $t \to \infty$. Since $\psi(\infty) = 0$, solution $x(t_0 + B, p(z_0))(t)$ tends to zero, i.e. is convergent. This solution, in view of construction, is a solution of Eq. (1) on interval I.

It remains to prove that every solution of Eq. (1) is convergent. Solutions $x(t_0 + B, p(z_0))$ and (initially assumed) $x(t_0, \chi)$ of Eq. (1) differ obviously (due to construction of the system of initial functions) by a constant (see Fig. 2). This means that solution $x(t_0, \chi)$ is asymptotically convergent, too. Since the solution $x(t_0, \chi)$ was arbitrarily taken, the proof remains valid for every fixed solution of Eq. (1). This remark completes the proof of $(\alpha) \Rightarrow (\beta)$.

4.5.2. The case $(\alpha) \Rightarrow (\gamma)$

This part is obvious, since it can be put $\nu(t) \equiv \Phi(t)$, where solution $\Phi(t)$ was defined in Section 4.5.1. The inequality (2) turns into an equality.

4.6. Proof of Theorem 4

In view of the validity of Theorems 1–3, it remains to prove only the case $(\gamma) \Rightarrow (\alpha)$. This part of proof uses topological principle and is similar to the reasonings in parts (a) and (d) of Section 4.5.1. Therefore we outline briefly only the main part of it. Suppose, without loss of generality (in view of condition (i)) that function ν is *positive* on I and $\nu(+\infty) = 0$.

4.6.1. Construction of a regular polyfacial set

Define the set

$$\omega_0^* \equiv \{(t,x):\ t \in I,\ x \in \mathbb{R}\}.$$

Except this, let us define the set ω^* in the same way as in the part (a) of Section 4.5.1 with

$$l_1(t,x) \equiv (x-\nu(t))(x+\nu(t)).$$

Then for $t \geqslant t_0$ (this interval is in accordance with the condition (β) of Definition 3):

$$Dl_1(t,x) = (f(t,x_t)-\dot\nu(t))(x+\nu(t)) + (x-\nu(t))(f(t,x_t)+\dot\nu(t)).$$

Estimate:

$$\begin{aligned} Dl_1|_{x=\nu(t)} &= 2\nu(t)\cdot(f(t,x_t)-\dot\nu(t))\big|_{x(t)=\nu(t)} \\ &\geqslant 2\nu(t)\cdot(f(t,x_t)-f(t,\nu_t))\big|_{x(t)=\nu(t)} \quad \text{(by (2))} \\ &> 2\nu(t)\cdot(f(t,\nu_t)-f(t,\nu_t)) = 0 \quad \text{(by (iv), } \varphi_t \equiv \nu_t) \end{aligned}$$

and

$$\begin{aligned} Dl_1|_{x=-\nu(t)} &= -2\nu(t)\cdot(f(t,x_t)+\dot\nu(t))\big|_{x(t)=-\nu(t)} \\ &= 2\nu(t)\cdot(f(t,-x_t)-\dot\nu(t))\big|_{x(t)=-\nu(t)} \quad \text{(by (ii))} \\ &\geqslant 2\nu(t)\cdot(f(t,-x_t)-f(t,\nu_t))\big|_{-x(t)=\nu(t)} \quad \text{(by (2))} \\ &= 2\nu(t)\cdot(f(t,z_t)-f(t,\nu_t))\big|_{z(t)=\nu(t)} \quad \text{(substitute } x \equiv -z) \\ &> 2\nu(t)\cdot(f(t,\nu_t)-f(t,\nu_t)) = 0 \quad \text{(by (iv), } \varphi_t \equiv \nu_t). \end{aligned}$$

Consequently, under the suppositions described in condition (β) of Definition 3: $Dl_1|_{(t,x)\in\partial\omega^*} > 0$. This means that the set ω^* is a regular polyfacial set with respect to Eq. (1).

4.6.2. Existence of increasing and convergent solution of Eq. (1)

Now we will proceed as in the part (d) of Section 4.5.1. Let us define the set W in the same way and let us put $Z \equiv A$ where

$$A = \{(t,x) \in \overline{\omega}^*,\ t = t_0,\ -\nu(t_0) \leqslant x \leqslant \nu(t_0)\}.$$

The construction of the set of initial functions p_{A,ω^*} is not done in detail here. We only remark that, in view of decreasity of ν, and from geometrical considerations, it is possible to construct this system consisting *only of increasing functions*. By analogy with the reasonings in Section 4.5.1(d), we conclude that there exists an initial function which generates the solution $x(t)$ of Eq. (1) with the property $(t,x(t)) \in \omega^*$ on I. So, the solution $x(t)$ tends to zero, i.e. is convergent. Since its initial function is increasing, the solution $x(t)$, in view of the property (v), is increasing too. So, the affirmation (α) is proved and this completes the proof of Theorem 4.

4.7. Proof of Corollary 1

This corollary is a consequence of Theorem 3.

4.8. Proof of Theorem 5

The strategy of the proof is to find an appropriate function λ^* which satisfies the inequality (8). Let us look for a function λ^* of the form

$$\lambda^*(t) = \frac{\varepsilon}{B_n^p(t) \cdot t \ln t \ldots \ln_n t}$$

where ε is a positive number. Then the left-hand side $\mathcal{L}(t)$ of inequality (8) becomes

$$\begin{aligned}
\mathcal{L}(t) = 1 + \lambda^*(t) &= 1 + \frac{\varepsilon}{B_n^p(t) \cdot t \ln t \ldots \ln_n t} \\
&= 1 + \frac{\varepsilon}{t \ln t \ldots \ln_n t} \cdot \left(\frac{1}{r} - \frac{1}{2t} - \cdots - \frac{1}{2t \ldots \ln_{n-1} t} - \frac{p}{2t \ldots \ln_n t}\right)^{-1} \\
&= 1 + \frac{r\varepsilon}{t \ln t \ldots \ln_n t} \\
&\quad \times \left(1 + \frac{r}{2t} + \cdots + \frac{r}{2t \ldots \ln_{n-1} t} + \frac{rp}{2t \ldots \ln_n t} + O\left(\frac{1}{t^2}\right)\right).
\end{aligned}$$

Furthermore, supposing $n \in \mathcal{N}$, we estimate (with the aid of Lemma 1 where $\sigma = -\varepsilon$ is put) the right-hand side of inequality (8):

$$\begin{aligned}
\mathcal{R}(t) = \mathrm{e}^{\int_{t-r}^{t} \beta(s)\lambda^*(s)\,\mathrm{d}s} &\leqslant \exp\left(\varepsilon \int_{t-r}^{t} \frac{\mathrm{d}s}{s \ln s \ldots \ln_n s}\right) \\
&= \exp\left(\varepsilon \ln \frac{\ln_n t}{\ln_n(t-r)}\right) = \frac{\ln_n^\varepsilon t}{\ln_n^\varepsilon (t-r)} \\
&= 1 + \frac{r\varepsilon}{t \ln t \ldots \ln_n t} + \frac{r^2\varepsilon}{2t^2 \ln t \ldots \ln_n t} + \cdots \\
&\quad + \frac{r^2\varepsilon}{2(t \ln t \ldots \ln_{n-1} t)^2 \ln_n t} + \frac{r^2\varepsilon(\varepsilon+1)}{2(t \ln t \ldots \ln_n t)^2} + o\left(\frac{1}{t^3}\right) \equiv \mathcal{R}^*(t).
\end{aligned}$$

Comparing the coefficients of identical functional terms of $\mathcal{L}(t)$ and $\mathcal{R}^*(t)$, we see that for $\mathcal{L}(t) \geqslant \mathcal{R}(t)$ it is sufficient that $p > \varepsilon + 1$. Since $\varepsilon > 0$ we get $p > 1$. Inequality (8) holds and, by Corollary 2, all the solutions converge. For $n = 0$, this part of the proof can be done similarly. The theorem is proved.

4.9. *Proof of Theorem 6*

As above, according to Corollary 2, it is necessary to choose an appropriate function λ^* which satisfies the inequality (8). Let us verify that the function $\lambda^* \equiv \lambda_n^\varepsilon(t)$ with a positive constant $\varepsilon \in (1, 1+\varepsilon_0)$ is such a function, since it satisfies all the conditions of Corollary 2. Without loss of generality, let us suppose that t_0 is sufficiently large. The following asymptotic developments are obtained with the aid of Lemma 1. Denote

$$\mathcal{L}(t) = 1 + \lambda^*(t) \equiv 1 + \lambda_n^\varepsilon(t)$$

and

$$\mathcal{R}(t) = \exp\left[\int_{t-r}^{t} \beta(s)\lambda^*(s)\,\mathrm{d}s\right] \equiv \exp\left[\int_{t-r}^{t} \beta(s)\lambda_n^\varepsilon(s)\,\mathrm{d}s\right].$$

Let us verify that $\mathcal{L}(t) > \mathcal{R}(t)$ for $t \to \infty$. Since the ratio $\lambda_n^\varepsilon(t)/\varphi(t)$ is a nonincreasing function we get

$$\begin{aligned}
\mathcal{R}(t) &\leqslant \exp\left[\frac{\lambda_n^\varepsilon(t-r)}{\varphi(t-r)} \int_{t-r}^{t} \beta(s)\varphi(s)\,\mathrm{d}s\right] \\
&\leqslant \exp\left[\frac{\lambda_n^\varepsilon(t-r)}{\varphi(t-r)} \cdot \varphi(t-r) L_n^p(t-r)\right] = \exp\left[\lambda_n^\varepsilon(t-r) L_n^p(t-r)\right] \\
&= 1 + \underbrace{\lambda_n^\varepsilon(t-r) L_n^p(t-r)}_{T_1(t)} + \underbrace{\frac{1}{2}\left[\lambda_n^\varepsilon(t-r) L_n^p(t-r)\right]^2}_{T_2(t)} + O\left(\frac{1}{t^3}\right) \\
&\equiv \mathcal{R}^\star(t).
\end{aligned}$$

Let us consider the expression

$$\begin{aligned}
T_1(t) = {} & \frac{1}{(t-r)\ln(t-r)\ldots\ln_{n-1}(t-r)\ln_n^\varepsilon(t-r)} \\
& \times \left[1 - \frac{r}{t-r} - \frac{r}{(t-r)\ln(t-r)} - \cdots - \frac{r}{(t-r)\ldots\ln_{n-1}(t-r)}\right. \\
& \left. \quad - \frac{pr}{(t-r)\ldots\ln_n(t-r)}\right].
\end{aligned}$$

Suppose $n \in \mathcal{N}$. Using Lemma 1 (with $\sigma = -1$ or $\sigma = -\varepsilon$) and the asymptotic representation of a fraction, we get

$$\begin{aligned}
T_1(t) = {} & \lambda_n^\varepsilon(t)\left(1 + \frac{r}{t} + \frac{r^2}{t^2} + O\left(\frac{1}{t^3}\right)\right)\left(1 + \frac{r}{t\ln t} + o\left(\frac{1}{t^2}\right)\right)\cdots \\
& \times \left(1 + \frac{r}{t\ln t\ldots\ln_{n-1} t} + o\left(\frac{1}{t^2}\right)\right)\left(1 + \frac{\varepsilon r}{t\ln t\ldots\ln_n t} + o\left(\frac{1}{t^2}\right)\right)
\end{aligned}$$

$$\times\left[1-\frac{r}{t}\left(1+\frac{r}{t}+\frac{r^2}{t^2}+O\left(\frac{1}{t^3}\right)\right)\right.$$
$$-\frac{r}{t\ln t}\left(1+\frac{r}{t}+O\left(\frac{1}{t^2}\right)\right)\left(1+\frac{r}{t\ln t}+o\left(\frac{1}{t^2}\right)\right)-\cdots$$
$$-\frac{r}{t\ldots\ln_{n-1}t}\left(1+\frac{r}{t}+O\left(\frac{1}{t^2}\right)\right)\left(1+\frac{r}{t\ln t}+o\left(\frac{1}{t^2}\right)\right)\cdots$$
$$\times\left(1+\frac{r}{t\ldots\ln_{n-1}t}+o\left(\frac{1}{t^2}\right)\right)$$
$$-\frac{pr}{t\ldots\ln_n t}\left(1+\frac{r}{t}+O\left(\frac{1}{t^2}\right)\right)\left(1+\frac{r}{t\ln t}+o\left(\frac{1}{t^2}\right)\right)\cdots$$
$$\left.\times\left(1+\frac{r}{t\ldots\ln_n t}+o\left(\frac{1}{t^2}\right)\right)\right].$$

After necessary computations we get

$$\begin{aligned}T_1(t)&=\lambda_n^{\varepsilon}(t)\left[1+\frac{r}{t}+\frac{r}{t\ln t}+\cdots+\frac{r}{t\ln t\ldots\ln_{n-1}t}+\frac{\varepsilon r}{t\ln t\ldots\ln_n t}+\frac{r^2}{t^2}\right.\\&\qquad\left.+o\left(\frac{1}{t^2}\right)\right]\\&\quad\times\left[1-\frac{r}{t}-\frac{r}{t\ln t}-\cdots-\frac{r}{t\ln t\ldots\ln_{n-1}t}-\frac{pr}{t\ln t\ldots\ln_n t}-\frac{r^2}{t^2}\right.\\&\qquad\left.+o\left(\frac{1}{t^2}\right)\right]\\&=\lambda_n^{\varepsilon}(t)\left(1+\frac{r(\varepsilon-p)}{t\ln t\ldots\ln_n t}-\frac{r^2}{t^2}+o\left(\frac{1}{t^2}\right)\right).\end{aligned}$$

Now it is easy to see that

$$T_2(t)\sim\frac{1}{2}\left[\lambda_n^{\varepsilon}(t)\right]^2=\frac{1}{2(t\ln t\ldots\ln_{n-1}t\ln_n^{\varepsilon}t)^2}.$$

Collecting the above asymptotic expansions we see that

$$\begin{aligned}\mathcal{R}(t)&\leqslant 1+\lambda_n^{\varepsilon}(t)+\lambda_n^{\varepsilon}(t)\left[\frac{r(\varepsilon-p)}{t\ldots\ln_n t}-\frac{r^2}{t^2}+o\left(\frac{1}{t^2}\right)\right]\\&\quad+\frac{1+o(1)}{2(t\ldots\ln_{n-1}t\ln_n^{\varepsilon}t)^2}\equiv\mathcal{R}^*.\end{aligned}$$

Comparing the coefficients of the identical functional terms of $\mathcal{L}(t)$ and $\mathcal{R}^*(t)$ we see that for $\mathcal{L}(t)\geqslant\mathcal{R}(t)$, the inequality $p>\varepsilon$ is sufficient. Since $\varepsilon>1$ (this assumption is necessary for the asymptotic dominance of the third term in above inequality), we get $p>1$. For $n=0$, this part of the proof can be done similarly. The theorem is proved.

4.10. Proof of Corollary 3

The inequality (14) follows immediately from the inequality (3).

4.11. Proof of Theorem 7

Let us put $L(t) = 1/t^p$ and suppose $p+q > 1$ and $q(1-\alpha) > p\alpha$. Such choice is in view of inequality $q > \alpha$ always possible. Under our assumptions, inequality (14) is valid if

$$L(t)m^\alpha \mathrm{e}^{-\alpha \int_{t_0}^t \beta(s)L(s)\,\mathrm{d}s} \geqslant \left(\mathrm{e}^{\int_{t-r}^t \beta(s)L(s)\,\mathrm{d}s} - 1\right)^{1-\alpha}$$

or, after a transformation,

$$1 + m^{\alpha/(1-\alpha)} L(t)^{1/(1-\alpha)} \exp\left[\frac{-\alpha}{1-\alpha} \cdot \int_{t_0}^{t} \beta(s)L(s)\,\mathrm{d}s\right] \geqslant \exp\left[\int_{t-r}^{t} \beta(s)L(s)\,\mathrm{d}s\right]. \tag{22}$$

Estimate the right-hand side $\mathcal{R}(t)$ of the inequality (22). We get (for $t \to \infty$)

$$\mathcal{R}(t) \leqslant \mathrm{e}^{C\int_{t-r}^t s^{-p-q}\,\mathrm{d}s} = 1 + \frac{Cr}{t^{p+q}} + o\left(\frac{1}{t^{p+q}}\right).$$

Estimation of the left-hand side $\mathcal{L}(t)$ of the inequality (22) gives

$$\mathcal{L}(t) \geqslant 1 + \frac{1}{t^{p/(1-\alpha)}} m^{\alpha/(1-\alpha)} \exp\left[\frac{-\alpha C}{1-\alpha} \int_{t_0}^{t} s^{-p-q}\,\mathrm{d}s\right] \geqslant 1 + \frac{1}{t^{p/(1-\alpha)}} m^{\alpha/(1-\alpha)} \mathrm{e}^K,$$

where K is an appropriate constant. Now for $\mathcal{R}(t) \leqslant \mathcal{L}(t)$ is sufficient

$$1 + \frac{1}{t^{p/(1-\alpha)}} m^{\alpha/(1-\alpha)} \mathrm{e}^K > 1 + \frac{Cr}{t^{p+q}} + o\left(\frac{1}{t^{p+q}}\right).$$

The last inequality holds, in view of the assumptions of the theorem, for $t \to \infty$. The theorem is proved.

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